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TRANSFORMATION GEOMETRY IN PRIMARY SCHOOL ACCORDING TO MICHEL DEMAL

Abstract. Michel Demal uses symmetries of the plane and the 3-D space to study plane and solid geometric figures. Transparencies which can glide and be flipped materialise motions and reversals in the plane plunged in the 3-D space. The extensive use of motions and reversals explains why Demal describes his program as "a transformation geometry course". He examines invariants by motions and reversals in the plane and in the space. He emphasises the importance of logic right from the beginning of his programme.

Résumé. Michel Demal utilise les symétries du plan et de l'espace pour étudier les figures géométriques du plan et de l'espace. Son utilisation intensive des déplacements et des retournements explique pourquoi Demal intitule son programme « géométrie des transformations ». Il étudie les invariants par déplacements et retournements dans le plan et dans l'espace. Il accorde beaucoup d'importance à la logique dès le début de son programme.

Keywords : chirality, direct, inverse genetic approach, geometry, logic, isometry, motion, orientation, plane, polygon, polyhedron, reversal, similarity, space, spiral teaching, transformation

Mots clés : antidéplacement, chiralité, déplacement, espace, approche génétique, géométrie, isométrie, logique, mouvement, orientation, plan, polygone, polyèdre, retournement, similitude, enseignement en spirale, transformation

Symmetry is a fascinating aspect of nature, but it is also a fundamental scientific concept which has invaded mathematics, physics, chemistry and even biology. Paul Valéry was perhaps thinking of symmetry when he wrote "There are no simple things, there is just a simple way of looking at things."
J. SIVARDIÈRE in [Siva]

1. The genesis of the project

Michel Demal who graduated in mathematics in 1975 has trained primary school teachers in mathematics for the last 28 years. His passion for geometry roots back to his student years at the University of Brussels when he met geometer Francis Buekenhout. Right from the beginning of his career in the teachers training college he embraced the project of teaching the geometry that matters to present day scientists to the young children of primary school, while maintaining sound mathematical foundations. Although many concepts had been known for a long time and are widely taught in the biology, chemistry or physics courses, they did not penetrate the world of mathematics in primary or secondary schools. The bilateral symmetry in animals and plants, the left and right symmetries in crystals – an object of research for Pasteur –, the chiral molecules that are so important in recent research were (and still are to a large extent) unknown to mathematics teachers.

The usual practice of geometry in primary schools is essentially observing and describing objects. Reasoning, building proofs are left to the secondary school. There is an enormous gap that children have to overcome when, at the age of 13, they are expected to write down proofs in a formal way. The next gap comes at University level where new science students painfully discover that the Euclidean geometry they sweated on is of little use to them. Michel Demal decided to fill these gaps.

In order to avoid preaching in the desert among primary school teachers, he took advice from his many former students who were now practicing in real classes with real children. He began experimenting his project in classes in 1984 and has not stopped since then[1]. In 1997, the project moved from small scale to a much wider and deeper when he started working in team with Danielle Popeler, an experienced and enthusiastic primary school teacher. She teaches geometry in various classes and they organise numerous training sessions for primary school teachers. He regularly discussed the mathematics underlying his project with Francis Buekenhout in order to keep his course towards living mathematics free of theoretical flaws. During one of these discussions, Francis Buekenhout said that the key to understanding was to be found in

[1] More details on the first steps of Demal's experiment can be found in [Étie] p. 47,48

the movements of objects in 3D-space. Demal first tried to work with the transformations in the 3D-space but he did not find a satisfying way of materialising transformations that inverted the orientation. And indeed it is not easy. We all know what it means to turn clockwise around a round-about and if we say a car is moving anticlockwise, we know it moves the other way round. We have a good image of the plane orientation. However, when it comes to our own hands, we know our left hand is different from our right hand and that the two hands are similar and yet we cannot physically take our left hand and transform it into a right hand. There is no elementary literature available on the topic of orientation. The University textbooks are too abstract, too technical, too difficult to be of any use. And yet nature is offering so many examples of it that there must be a way of explaining the orientation.

Demal resolved to begin with transformations of the plane before examining the 3D-space. He had found a remarkable way of materialising the transformations with a transparent sheet (sometimes called a slide) he could glide (without lifting it), rotate (without lifting it), and lift and turn in space and place it back on the flat surface. This led him to distinguish two kinds of transformations: the “motions” (or idealised movements in French *déplacements*) which kept the orientation unchanged, and the “reversals” (in French *retournements*) which inverted the orientation. The “motions” include translations and rotations, and did not require to lift the transparent sheet, while the “reversals” including the reflections and the glide reflections required one lift and turn. He had in mind the symmetries of an object. Remember the wooden puzzles of your childhood where you had to put a little animal of a given shape back into the right hole. There was a little nail in the middle of the piece, so that you could easily grip and turn it. With a square piece, you could place it back in its hole in four different ways. This is precisely what symmetries are all about. In how many ways can you transform an object into itself. The puzzle model is good enough to materialise “motions“ but it shows its limits for the “reversals”. You might drop the nail, but the piece is not painted at the back, it does not look the same. The transparent sheet is ideal to materialise both the “motions” and the “reversals” in the plane. Using the mathematical jargon, Demal is introducing the *automorphism* (from Greek auto = self, morphism = transformation), i.e. transformation that maps an object into itself or, more simply symmetry. He is fond of the technical word because most of us, when hearing the word symmetry, think of “mirror symmetry” and don’t include translations and rotations. The maths teachers would use another technical word to describe the symmetries of an object. They would speak of *isometry* (from Greek iso = same, metries = measure) i.e. transformation that does not alter distances. He also had in mind the enlargements and the reductions of objects. He normally uses his hands to show an enlargement, but there are many tools to materialise enlargements and reductions like the pantograph or the photographic enlargements. Demal was thinking of the similarity in the plane.
He first considers finite objects in the plane and in the 3D-space, but he has in mind the automorphisms of the plane and of the 3D-space.

For a long time, Demal used cardboard hands with nails drawn at the tip of the fingers to materialise the orientation of the plane, but somehow the concept was not accepted by the children. Whenever he turned the cardboard of a hand, the children had the palm of the hand in mind and not a right hand reversed into a left hand. Thanks to Danielle Popeler, he abandoned the cardboard hands for hands drawn on a transparent sheet. They are careful to use the expression “*drawing of hand*” (in French “dessin de main”) when they speak of the orientation of the plane. Thanks to the knowledge acquired in the plane, the orientation in the 3D-space is materialised by a 3D-hand. A transformation of the 3D-space which keeps the orientation unchanged, changes a right hand into a right hand and a transformation which inverts the orientation changes a right hand into a left hand.
The extensive use of motions and reversals explains why Demal describes his program as “a transformation geometry course”. He examines invariants by motions and reversals in the plane and in the space. He approaches direct and inverse similarities in the plane and in the space. He analyses the orientation of objects (chirality). He classifies regular-face convex polyhedra according to homogeneity of faces and vertices. [Dema 5].
Demal is still busy experimenting with space transformations in the 6^{th} grade and he is planning to have a geometry course for children of 5 to 14 years.
One of the strong points of Demal’s project is the mathematical coherence. Buekenhout has analysed the foundations of Demal’s transformation geometry in [Buek 7].

2. The project itself

The last twenty years have seen many restructurings in mathematics teaching in primary and secondary schools resulting in a definite shrinking of geometry in the curriculum. It is relevant to ask ourselves what mathematics and in particular what geometry we would like our children to master.
In 1982, at the International Colloquium on Geometry Teaching, Buekenhout wrote what we can call the founding text for Demal's project [Buek 6] from which I shall select several quotations.
Commenting on Freudenthal's quotation:

> *"…Geometry is grasping space. And since it is about the education of children, it is grasping the space in which the child lives, breathes and moves, the space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it."*

Buekenhout adds:

"

- *geometry for space and space for geometry*
- *the (Euclidean) space by and for its main subsets, namely cylinder, parallelepiped, cone, rectangle, prism, circle, sphere, etc.*
- *through its main motions, namely rotation and translation*[2]*,*
- *through reflections, similarities, projections,*
- *through number and measure."* [Buek 6]

Further in the text we find the guidelines of Demal's project:
About figures:

> *"Euclidean space is not available at the bottom level. What is available instead, is a collection of figures, actually an incredibly rich collection of figures in which some order has soon to be established. Which order? It is quite natural to distinguish the most important figures (P. Libois) namely those which are most often observed around us. They are: cylinder, parallelepiped, cone, rectangle, circle, sphere, prism, segment of line, etc. All figures are bound at the age of 12 and it may be good to realise that this was the case for Euclid too. The study of these figures and of their motions, particularly the cylinder, leads to translations which arise also from ornaments and their construction by repetition. It leads to more figures such as polygons, polyhedra, planes, conics, helices, tessellations, etc. The needs of simplification (of translation and motion) and of unification (of the fantastic number of figures) leads to unbounded figures such as unbounded cylinders and cones, lines, half-lines, planes, half-planes and finally, a figure containing all others, one which achieves the highest degree of symmetry since it is invariant by each motion, one in which translations and all motions become perfect, every point having an image and being the image of another."* [BUEK 6]

About transformations

> *"One hears quite often that figures are more natural to children than transformations. If we think in bottom level terms this is ridiculous. Movement is as familiar as figures and neither exists nor would exist without support from the other. What is true, in fact, is that the children's attention is drawn to figures by education, much more than to movement. This does not mean that there is a necessity here."* [Buek 6]

About orientation:

> *"Whose duty would it be to explain the subtle difference between right and left if it is not the geometry teacher? This requires some developments on orientation, which are short and easy when the transformations of space are available and on a much lower level than in terms of determinants."* [Buek 6]

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[2] Buekenhout's vision has changed. Today, he would not insist on translation. Physically, a translation cannot be distinguished from a rotation whose axis is far away.

More than twenty years after the first experiments in classes, let us see how Demal describes his program:

> *"…One should not just discover properties of geometric objects and properties of transformations independently. One should link them in order to connect properties and learn gradually how to prove.*
> *The concept that enables to link the properties of objects to the properties of transformations is called symmetries or automorphisms which means transformations which transform an object into itself while respecting its structure. We call the geometry that links the properties of transformations to the properties of objects "transformations geometry" and we can characterise it as the study of geometric objects in the plane and in 3 dimensional space using the transformations.*
> *Transformations are tools whose properties enable to*
> - *discover and/or prove properties of geometric objects;*
> - *create regular figures (patterns) like friezes, rosettes, wallpapers;*
> - *classify geometric objects;*
> - *perceive the chirality of an object."*
>
> http://www.uvgt.net/ :La philosophie [Dema 1]

Demal begins to classify geometric objects (he distinguishes figures in the plane and solids in 3D-space) right from the first year primary school. The criterion emanates from the children themselves. They speak of "curved sides". Note that this leads to a non-trivial classification and the practice of logic at a very elementary level. There are three classes of plane geometric figures: the set of plane geometric figures (PGF) for which all sides are straight segments of line (in French "côté droit"), the set of hybrid plane geometric figures for which some sides are straight and some sides are curved and the set of curved PGF for which all the sides are curved. He then defines the polygons. A more subtle classification takes place for the solid geometric figures (SGF). An SGF for which all the faces are polygons is a polyhedron. The two other classes of SGF are the curved solids and the hybrid solids.

Concerning transformations, Demal is directly inspired by Buekenhout and the motions. For the plane geometry he uses the transparent sheet to materialise the motions: this quickly leads him to translations and rotations. By turning the transparent sheet upside down he obtains another set of transformations: the reversals. Doing this he has all the isometries and a very powerful tool to study the orientation of PGF. He also considers transformations that enlarge or reduce and henceforward the similarities. He adds the parallel projections (materialised by the solar shade).

Demal decided to approach solid geometry after plane geometry because, as mentioned earlier, he had difficulties in materialising reversals in solid geometry. One obvious transformation is the mirror symmetry or bilateral symmetry or reflection for which Demal uses mirrors, glass doors and psychomotility exercises in order to perceive the left and right orientations in 3D space. However, when it comes to the problem of finding which transformation of the solid space could transform a left hand in any position into a right hand in any position, children and many adults, even those who are highly trained in mathematics, are blocked and do not understand the two orientations in the 3D space. Let us analyse the problem of the 3D space, the space in which we live and move ourselves, the space of our perceptions[3]. Mathematically, right and left are undistinguishable. And, let's face it, for many of us they remain undistinguishable even when we are grown up. But nature made a choice. Amino acids found in nature are left oriented. Pasteur discovered that crystals of mould in casks all had the same orientation but that he could synthesise crystals of the other orientation in his lab.

> *"A reference object like the human body can include a left-oriented molecule and fall seriously ill when absorbing the corresponding right-oriented molecule. The sinister thalidomide is a well-known example. The mad cow disease whose explanation is linked to chirality led Stanley Prusciner to the Nobel price of medicine in 1997. "* [Buek 3]

As structures get more complex, left and right can be distinguished and it is not because some of us suffer from orientation blindness that orientation should not be analysed and taught in geometry classes. Is it

[3] The paragraph on orientation has been written after a conversation with Francis Buekenhout who is the inspirer of the text. More details can be found in [Buek 13] and [Buek 14].

reasonable to leave it to the military to teach how to read a map and orient oneself on the ground? There are subtleties of the orientation that are not perceived by mathematically trained adults who have been flattened out by education and refer preferably to plane geometry. Most maths teachers in the secondary school are not aware that a "point symmetry" or "central inversion" or "reflection in a point", which transforms each point P into the point P' for which the midpoint of PP' is a fixed point O ([Coxe] p.98), does not function the same way in the Euclidean plane and in the Euclidean space. In the plane, a point symmetry is a half-turn rotation which does not alter the orientation. In the 3D space, a central inversion is not a rotation, it has only one fixed point and it does invert the orientation.

Imagine a glass door on each side of which there is a child. When one child draws a circle clockwise on his side of the door, the other child sees a circle drawn anticlockwise. Mathematically, this means that turning right or left in a plane depends on which side of the plane you stand.

To decide whether an object in space is left or right or both we can use the following criterion: if the object can be transformed into itself only by motion in space then the object has an orientation. It is right or it is left. But if the object can be transformed into itself by a motion and by a mirror-symmetry then it has no orientation. It is both right and left[4].

In our body, we know the difference between right and left, because we stand vertically on the ground, because our head is different from our feet. When we shake hands with a neighbour we associate our right hand with the right hand of our neighbour. A left-handed person wanting to cut a piece of paper associates his left hand with a left-handed pair of scissors. We know whether a spiral staircase turns to the right or to the left. But for an object in space like a satellite, assuming it is not submitted to translation, a rotation to the right cannot be distinguished from a rotation to the left.

We are very familiar with the shape of a corkscrew and to uncork a bottle we combine a rotation to the right and a translation. We make a twist [Coxe]. If we decide to uncork the bottle by placing the bottle on top of the corkscrew, we would still turn to the right. The orientation of a twist is absolute.

One should credit Demal for having analysed these difficulties in depth and developed a battery of exercises to approach the orientation both in the Euclidean plane and the Euclidean space. A reversal in the plane transforms the drawing of a left hand on a transparent sheet into the drawing of a right hand. A reversal in the 3D space transforms a left hand into a right hand.

Another remarkable aspect of Demal's program is the introduction of logical reasoning at a very early stage.

> *"Demal and his team practice proof with 6 year old children and above, systematically, during the whole school year, from the first until the sixth primary school year. They are playing games, mathematical games conceived and created for geometry. It is a rich, unbelievable but true experience. They necessarily adopt a genetic vision of proofs with a spiral development. Demal says nothing is free but that everything can be explained and taught. Small collective proofs involving everyone in the class, using material, not attempting a formal writing but establishing gradually the basis of logic."* [Buek 11]

Demal remembers the answer Jacques Tits gave to the question "What is mathematics?" "At any stage, doing mathematics is working while respecting the rules of logic". In order to go forward in mathematics, one has to understand the rules of the game of logic. The understanding is not innate. At the beginning of secondary school, children are expected to write down proofs respecting the rules and conventions of logic and most of them do not understand the rules of the game they are supposed to play. There should be an intuitive approach outside mathematics before introducing mathematical concepts.

Let us see how he approaches the universal quantifier in the first year primary school. The criterion used to classify Plane Geometric Figures (PGF) is based on "straight" sides and "curved" sides. A PGF all of whose sides are straight is a polygon, a PGF all of whose sides are curved sides is a "round" PGF and a PGF having both straight and curved sides is a "hybrid" figure. This classification allows one to speak of "at least one curved side". At a later stage, Demal approaches the negation of the universal quantifier by classifying PGF in polygons and non-polygons, a non-polygon being a PGF for which not all sides are straight. This can be translated into "a PGF for which at least one side is curved".

[4] A subtle consequence of this is that any finite plane object plunged in space is necessarily both right and left. The bilateral symmetry whose plane of fixed points is the same as the plane containing an oriented circle plunged in the 3D space transforms this oriented circle into itself.

In English or French, coordinating conjunctions like *and* or *or* have several meanings whereas there is only one meaning accepted in mathematics. The ambiguities of language must be detected thanks to a few examples taken outside the field of mathematics before approaching the conjunction *and* and the disjunction *or* in logic. Consider sentence 1: "Students wearing glasses and students wearing a scarf are allowed to go out" and sentence 2: "Students wearing glasses and a scarf are allowed to go out". In the case of sentence 2, only students wearing both glasses and scarf are allowed to go out, whereas students need only one of them to be allowed out. In geometry, Demal trains pupils with logical propositions like: "A quadrilateral having two pairs of parallel sides *and* one right angle belongs to the family of rectangles". The spiral teaching of logic helps children to master two common difficulties:

- Children are blocked at the first part of the proposition and consider that only "true" parallelograms (not rectangles) are acceptable.
- Children reject the proposition because they consider a rectangle has *four* right angles, not just one.
- The coordinating conjunction *or* has three different meanings in English or French:
- The exclusive *or*: "He comes in *or* he goes out",
- The inclusive *or*: "Students wearing glasses *or* a scarf are allowed to go out",
- The equivalent *or*: "Bruges *or* Venice of the North".

In logic, the disjunction *or* is inclusive. For example, all the sides and all the angles of a regular polygon are isometric. A non-regular polygon is a polygon for which not all sides or not all angles are isometric. Take the case of regular quadrilaterals. A regular quadrilateral (a square) has four isometric sides and four isometric right angles. A non-regular quadrilateral can be a rectangle with two adjacent sides of different length and four right angles. It can be a rhombus with two angles of different measure and four isometric sides. It can be a kite and it can be an irregular quadrilateral.

In grade 4 (age 9-10), Demal introduces the ab absurdo proof. Children first build several polyhedra with regular triangular faces (tetrahedron, octahedron, icosahedron, deltahedra). The children first count the number of straws they need to build one face and then multiply that number by the number of faces. They build the triangular faces and assemble them. Each edge of the polyhedron contains two straws and they are asked to compute the number of edges according to the number of straws (e.g.: tetrahedron, $3 \times 4 = 12$ straws, $6 = 12/2$ edges). Children are then asked how many straws they would need to build a polyhedron with 5 regular triangular faces. With $3 \times 5 = 15$ straws they can build 5 triangular faces. Each edge of the polyhedron consists of 2 straws coming from the adjacent faces. The number of edges of the polyhedron is necessarily the number of straws divided by 2. Hence, if the number of straws is odd, the number of edges is not an integer number and a polyhedron with 5 regular triangular faces cannot exist. A polyhedron with regular triangular faces has necessarily an even number of faces.

The logical implication is absolutely fundamental to understand the nature of proof but it is out of the question to introduce the notion with a truth table. Children should first realise that there are causality relations and dependency relations between elements that seem, at first, unrelated. In quadrilaterals, there is a relation between parallelism and isometric angles. They should get familiar with the notion of true consequence and false consequence under the hypothesis of a true premise. Experience and intuition lead us to admit that the logical proposition $p \Rightarrow q$ is true when p is true and q is true, that $p \Rightarrow q$ is false when p is true and q is false. However, it is much harder to admit that $p \Rightarrow q$ is true when p is false, whichever truth-value q might have. We have to accept the two following implications

$3+4=9 \Rightarrow$ Brussels is the capital of France

$3+4=9 \Rightarrow$ Brussels is the capital of Belgium

This requires a sense of humour only accessible to logically trained people.

Normally, beginners are asked to prove theorems (implications) with true hypotheses (premises). There is a difference between the inference $\Rightarrow$ the "physical causality" and the implication $p \Rightarrow q$. Demal has a whole battery of exercises on implication. Children tend to confuse $p \Rightarrow q$ and $q \Rightarrow p$, because the two propositions seem identical and express the same idea. For example, he asks which of the following propositions are true:

- There is no more petrol in the petrol-tank $\Rightarrow$ the engine stop
- The engine stops $\Rightarrow$ there is no more petrol in the petrol-tank
- The quadrilateral has four right angles $\Rightarrow$ the diagonals of the quadrilateral are of equal length.
- The diagonals of the quadrilateral are of equal length $\Rightarrow$ the quadrilateral has four right angles.

The most up-to-date vision of Demal's program is summarised on his website http://www.uvgt.net/ section "Aspect théorique", 5. Les objectifs de la géométrie des transformations à travers le fondamental [Dema 1].

"The aims

- *First, encourage the acquisition by children of a certain number of theoretical concepts on families (sets) of geometric objects of the plane and the 3D space, by means of observation, manipulation, construction, comparison, analysis, drawing. Children should also encounter "situations-problèmes (in French)" so that they would discover how to conjecture, to question propositions, to try to refute or accept them.*
- *Second, accustom the children to use transformations in the plane and in the 3D space in order to discover and/or justify the properties associated to plane and solid geometric objects.*
- *Third, enable children to imprint a mental image of the most common transformations.*
- *Fourth, show that there is a continuity in the methods, the way of thinking and the concepts developed in the primary and the secondary school. Show that there is no gap in the curriculum of compulsory education and favour a mental image and an in-depth knowledge of the subjects thanks to the principles of the genetic approach and the spiral teaching.*
- *Fifth, accustom children to geometric concepts common in science today.*

Our course of transformation geometry (of the plane and the 3D space) comprises the following innovative features:

- *a genetic geometry;*
- *based on spiral teaching*
- *not limited to situations of daily life*
- *guiding children towards a first approach of proofs*
- *introducing theoretical concepts that are usually not tackled in primary school:*
    1. *Accustom primary school children to the concepts of motion and reversals of figures before expressing these as reflections, translations, rotations or point symmetry.*
    2. *Make use of the invariants of motions and reversals in the study of geometric objects.*
    3. *Make use of the automorphisms or symmetries that superimpose a figure to itself: in a first step, this concept is not expressed in terms of reflections, rotations or translations.*
    4. *Make use of the orientation of the plane (circles oriented clockwise or anticlockwise and hands drawn on a transparent sheet) to distinguish motions from reversals in the plane.*
    5. *Make use of hands to distinguish motions from reversals in the 3D space.*
    6. *Classify convex polyhedra according to faces and vertices regularity (regular polyhedra, semi-regular polyhedra)."*

We shall see in section 3 how Demal succeeds in making transformation geometry accessible to young children by organising a genetic spiral teaching. Section 7 will be devoted to the perception of a professional mathematician on Demal's project. Francis Buekenhout has watched the project from birth to maturity and he can determine how far Demal succeeded in achieving to introduce concepts that matter to a present day scientist while keeping theoretical standards compatible with present day mathematics.

3. THE FOUR GENETIC SPIRALS

Demal's project is accessible and stimulating for the children who take part in the program. Danielle Popeler is extremely cautious in her approach to geometry with children. She carefully plans the lessons according to the age and experience of the children, never introducing a concept that is not accessible to young children and this explains why the program is convincing and successful. Right from the beginning, Demal followed two pedagogical principles: spiral teaching with a genetic approach.

3.1 Spiral teaching

Buekenhout is one of the first Belgian mathematicians to have popularised Jerome Bruner spiral curriculum [Brun]. However it is Buekenhout's interpretation of Bruner which prevails in Belgian didactical circles [CFWB].
Buekenhout in [Buek 9] chapter 1, has developed his vision of spiral teaching:

"The spiral? Or rather a helix in space. It symbolises the approach of a student, maybe guided by a teacher or a book in order to master a notion. It is opposed to the idea of a linear teaching, going directly to the target but essentially dogmatic, pitiless for the weak and absent-minded students, based on memory and on a flawless organisation. The spiral (the helix) combines the linearity of the progression and of the target, with the necessary revisions and questioning of previously acquired notions, in the light of newly acquired notions."

In [Buek 10] he adds:

"A mathematical notion and its spiral should be seen as a living tissue that everyone nourishes by brain work. This tissue comes to life, develops itself, ramifies. Some parts die, some are dormant or fade in favour of more efficient ones. The tissue interacts with other spirals built up by the same brain and with analogous spirals built up by others."

Demal follows the three principles put forward by Buekenhout in [Buek 6] for his spiral teaching:

Principle 1: The topics we decide to teach should be of permanent interest for all students throughout their life and the most important topics should appear as soon and as often as possible.

Principle 2: The main topics of interest to the student, deserve being studied several times, with each new exposure including both new approaches and a higher level of sophistication.

Principle 3: The study should start at a bottom level where the concepts are manipulated by the pupil but where he does not know what he is doing. However the child should not be directed to train its mathematical abilities on the bottom level, unless he is, in principle, able to progress to the next level, which means able to reflect on his bottom level mathematical abilities.

3.2 The genetic principle

"The genetic viewpoint is based on the idea that mathematical notions come to life, develop both in the history of mankind and in the history of individuals. That their coming to life is preceded by maturation. That it is important to retrace and understand the steps of this development in order to help all the persons interested in their progressive mastering of notions. That one should stop considering mathematics and proofs as frozen end products"

[Buek 11]

Erich Wittmann who had a most important influence on Demal's vision of the genetic approach wrote in [Witt 2]

"Mathematics teaching is doing mathematics with students in order to cultivate their understanding of reality."

Demal follows the principles Wittmann wrote about in [Witt 1]:

"The genetic method should:

- *refer to the previous knowledge of the learner*
- *encapsulate arguments in global problem context outside and within mathematics*
- *allow informal introduction of concepts from the context*
- *introduce more rigorous arguments starting from intuitive and heuristic elements, while maintaining motivation and continuity*
- *during this process, gradually enlarge the horizon and adapt the corresponding points of view."*

In the talk Wittmann gave in Nivelles in 1994 [Buek 12], he advocated to:

"

- *conceive a coherent and global curriculum;*
- *cover a list of basic notions with a good command of skills;*
- *include proofs accessible to pupils and teachers;*
- *teach within the granted time;*
- *without adding an extra burden on teachers."*

3.3 The four genetic spirals

With these guidelines in mind Demal has developed four interwoven genetic spirals:

- one for the geometric objects
- one for the transformations

- one for the first elements of logic
- one for the scientific approach.

They can be discovered by reading the curriculum carefully.

4. The Curriculum

The curriculum in French including pictures that illustrate the various activities can be found on the website [Dema 1] under the item “plans du cours”. An English version of the curriculum is available in [Dema 3]. Danielle Popeler has conducted the experiment in the primary schools from class 1 to class 4. She has conceived, tested and written the lessons, having in mind the fact that primary school teachers are “general practitioners” and not maths specialists. Of course she also had pupils in mind: those who changed school or were absent for a long time. They can catch up easily thanks to the spiral teaching. There are about 10 themes that are revisited, extended and deepened every year in the spirit of the spiral teaching: Plane geometric figures, solid geometric figures, transformations of the plane, length of a side, angles, straight lines, classification of polygons according to the number of sides, characteristics of plane geometric figures: the quadrilaterals: squares, rectangles, rhombi, parallelograms, trapezoids, disks and circles, triangles, friezes, rosettes.

Grade 1 (age 6-7)*:* Plane geometric figures. Solid geometric figures. Transformations of the plane (motions and reversals). Lengths. Angles. Straight lines. Horizontal straight lines. Vertical straight lines (materialised by plumb lines). Parallel lines. Classification of polygons according to the number of sides. First characteristics of plane geometric figures. Quadrilaterals: squares, ordinary rectangles. Friezes.
Grade 2 (age 7-8)*:* Closed plane geometric figures. Solid geometric figures. Transformations of the plane. Similarities of the plane. Similarities in plane and space. Lengths. Angles. Classification of polygons according to the number of sides. First characteristics of plane geometric figures. Quadrilaterals: squares, ordinary rhombi, ordinary rectangles, ordinary parallelograms. Triangles. Friezes.
Grade 3 (age 8-9)*:* Closed plane geometric figures. Solid geometric figures. Transformations of the plane. Similarities of the plane. Angles. Classification of polygons. Relative positions of two straight lines in the plane. Families of quadrilaterals: family of squares, family of rhombi, family of rectangles (discovery of automorphisms in terms of motions and reversals). Comparison between the three families mentioned above. Family of parallelograms. Friezes.
Grade 4 (age 9-10)*:* Closed plane geometric figures. Solid geometric figures. Transformations of the plane. Similarities of the plane. Angles. Straight lines. Segments of straight lines. Midpoint of a segment. Orientations of the plane. Circles. Rotations in the plane (including clockwise and anticlockwise rotations). Reflections in the plane. Families of quadrilaterals (extension of grade 4 curriculum): family of squares, family of rectangles, family of parallelograms, family of trapezoids (automorphisms are expressed in terms of rotations or reflections). Rosettes.
Grade 5 (age 10-11)*:* Closed plane geometric figures. Solid geometric figures. Transformations of the plane. Straight lines. Segments of straight lines. Angles. Relative positions of two straight lines in the plane and in the space. Circles. Disks. Specific motions. Specific reversals. Convex quadrilaterals. Classification in subfamilies: squares, rectangles, rhombi, parallelograms, trapezoids, ordinary quadrilaterals (discovery and proof of the properties of diagonals and medians with the help of automorphisms). Triangles. Classification in families and subfamilies: equilateral triangles, isosceles triangles, right-angled triangles, ordinary triangles.
Grade 6 (age 11-12)*:* extension of grade 5 curriculum. Plane geometry: triangles, compositions of motions and/or reversals, homotheties [Dema 5]. Solid geometry: motions transform a left hand into a left hand, reversals transform a left hand into a right hand. Study of polyhedra and of “round” solid figures.
We shall examine in details the genetic spiral of the quadrilaterals in section 5.

5. The genetic spiral of the quadrilaterals

In the first year, quadrilaterals are studied individually before grouping them into “families” and “subfamilies” having common characteristics. The first quadrilateral to be studied is the square: the “most

beautiful" quadrilateral. Children have to recognise squares in a collection of quadrilaterals. They compare the length of the four sides and realise that measuring one side is sufficient. They discover the two pairs of parallel sides thanks to two vertical plumb lines. They build squares with meccano rods and with parallel lines drawn on transparent strips. They discover the determining conditions of the square: four isometric sides, four right angles and they compare with rhombi and rectangles.
The next figure to be examined is the rectangle. Children discover the characteristics of the rectangle by comparing them with those of the square. They construct rectangles with coloured transparent strips and discover at the same time the rhombi and the parallelograms. They are asked to select potential rectangles in a collection of meccano assembled quadrilaterals. They compare rectangles to trapezoids.

In the second year, after having revisited the first year discoveries, children are then asked to select appropriate assembled meccano rods that could give squares. They discover they can distort a rhombus into a square. They then compare the lengths of the sides and they check that a square has four right angles. They detect squares on solid geometric figures. They build cubes. They then study rhombi and discover that squares are rhombi. This is a fantastic opportunity to practice logic at an early age. For the study of rectangles, they first select and they build rectangles. They select potential rectangles in a collection of quadrilaterals and discover the parallelogram. They distort a parallelogram assembled with meccano rods into a rectangle and discover that just one right angle is enough to give a rectangle. They detect opposite angles on rectangles. They are ready to study the parallelograms. They first select the parallelograms in a collection of quadrilaterals. They detect and compare opposite angles using angles drawn on transparent strips

In the third year, Children are grouping quadrilaterals into "families" and "subfamilies" according to common characteristics. They proceed from inside to outside, i.e. they start with the square, then they move to the rhombi, then the rectangles, then the parallelograms and finally to convex quadrilaterals. After having rediscovered the common qualities of all the squares, they examine the symmetries of the square: A square can be superimposed on itself by motions and by reversals. For the rhombi, the rectangles and the parallelograms, they examine the symmetries, distinguishing the motions and the reversals. They compare squares, rhombi, rectangles and parallelograms and obtain a Venn representation.

In the fourth year, after having studied reflections and rotations they analyse the symmetries of the square in detail. They first examine the two notions of diagonal of a quadrilateral and line joining the midpoints of opposite sides of a quadrilateral. Then examine how many different motions apply a square onto itself: rotations of 90°, 180°, 270°, 360° about the centre of the square and how many different reversals: the diagonals and the lines joining midpoints of opposite sides of a square are the axis of reflection. They insist on the use of reflection of the plane containing the figure instead of folding of the figure onto itself. The same analysis is done for rhombi, rectangles, parallelograms and trapezoids.

One should notice the importance of psychomotility, of the manipulation of various materials before approaching more abstract representations. The logical concepts are approached right from the start, in an informal way: the genetic spiral of logic can be found in most of the exercises.

6. A mathematician's point of view

Francis Buekenhout has recently analysed the foundations of Demal's transformation geometry in [Buek 7]. Most of the quotations in this section come from this article.

> "*Michel Demal has built a coherent and detailed curriculum of so-called Transformation Geometry for children aged 6 to 12. Enormous didactical efforts have been made by Demal and various co-workers especially Danielle Popeler over many years.*" [Buek 7]

Before starting the analysis, Buekenhout examines the nature of geometry.

> "*A key point is the relation to Physics. Most Geometry taught in schools is necessarily based on physical objects, their study, observation and manipulation. This may occur even at the level of genuine creation and research.* " [Buek 7]

"*In my opinion, Demal's Physical Geometry as developed in classrooms and texts requires a Mathematical foundation*." [Buek 7]

In the paragraph "on geometry" Buekenhout quotes [Buek 1]

"Geometry has its roots in the physical, visual and muscular perception of spatial objects. Its main historical sources are Euclidean geometry and projective geometry. The first arises from an idealised observation of solid bodies in motion while the second similarly derives from the optical one-eye observation of solid bodies. At our level of axiomatised or constructive mathematics, geometry appears as a study of structured sets with spatial inspiration or inspiration from former geometric subjects. Geometry studies spaces. A space can be seen as a structured set whose elements are called points. Typical examples are the usual (i.e. Euclidean) space, plane and line of elementary mathematics. The latter space structure, as a model of physical space, is not unique. Non-Euclidean geometry provides a classical illustration of this fact."

He then examines the primitive concepts for geometry according to Demal.

"As I understand it, the pillars of Demal's Geometry are space, figures and transformations.." [Buek 7].

On plane and space he writes:

"Demal's Geometry develops itself inside an object called Space which is assumed to be known without any further explanation. This is typically a physical approach. It is classical. When I tell undergraduate students about many mathematical spaces there is always one who comes to me afterwards and invokes the "real" Space." [Buek 7]

"In a first stage Demal deals with Geometry in a plane extended by space. This means that transformations of the plane may require a surrounding space. You may turn the plane upside down. Let me call this approach "plane in space".

"When he moves to theoretical work on solids no longer included in the plane, Demal requires Plane Geometry in order to explain Spatial Geometry. This is classical but rather annoying to my eyes acquainted with geometry-algebra independent of dimension".

"I should underline the courage and scientific clearness of sight of Demal. Most taught physical Geometry occurs in dimension two but everyone uses dimension three effectively anywhere outside Geometry and so nobody takes Geometry very seriously."

On figures he writes:

"A great quality of Demal's figures with respect to standard teaching is that the notion is remarkably general and open. It is not limited to a few objects. It fits with the true needs of science and culture. It is not the case however that Demal points to mathematical subtleties such as sets dense everywhere, fractals and in his practice every figure is bounded up to exceptions such as lines, planes, space."

On transformations he writes:

"Transformations as in Demal's Geometry are not analysed as a mathematical being. They are automorphisms of space for a structure which is slightly variable. They may be isometries, similarities, affinities. They apply to figures either to transform one in another or to transform in itself. They very much play a role as morphisms. However, their composition is not used. Transformations such as rotation, translation, reflection, dilatation are used constantly as the major tool but they are not studied for themselves independently of figures."

This article goes further in a very thorough analysis of the foundations of Demal's geometry that goes far beyond the scope of my presentation of Demal's project.

7. Conclusions

According to Joëlle Lamon, who analysed the Belgian results of the PISA 2000 survey, primary school teachers are uncomfortable in teaching geometry. They tend to avoid the subject in the first and second grade and they hardly ever approach solid geometry. The dominant textbook [Roeg], which is not free from errors, has a great influence on teaching. So far, in most classes, geometry lessons consist of observing, describing, drawing figures and geometric objects, drawing images of figures submitted to translations, rotations, reflections. Logic and reasoning are absent.

The various research projects on elementary geometry in primary school in the French-speaking part of Belgium are not supported by statistical evidence.

One interesting project on a “nail-board” was conducted by Pierre Marlier [Marl]. Marlier has devised a tool to create geometric figures and make some reasoning on them.

There are very few textbooks on geometry that are accessible to primary school teachers.

Geometry, An Investigative Approach [O’Daf], published in 1977, is still innovative and very helpful to teachers. It provides discovery experiences.

The excellent *Discovering Geometry, An Inductive Approach* [Serr], has a remarkable way of introducing logic. It is a genetic approach.

Unfortunately, most primary school teachers and math teachers in the French-speaking part of Belgium do not master English well enough to benefit from these textbooks.

André Deledicq’s *Maths Collège* [Dele] also has an interesting introduction on logic.

The demand for good textbooks and ready-made lessons on geometry is very high among primary school teachers. However the gap between “thinkers” and “doers” is very wide. As Wittmann states in [Witt 3]

“For systemic reasons, it is highly unlikely that theories that have been developed independently of practice can be applied afterwards. “The developing theory of mathematical learning and teaching must be a refinement, an extension, and deepening of practitioner knowledge, not a separate growth*”, as stated by Alan Bell in the mid-eighties [Bell].”*

Michel Demal and Danielle Popeler aim at helping primary school teachers to teach geometry. The course is intended to introduce children to important contemporary scientific concepts. Demal devised the theoretical frame and Popeler conceived, experienced and wrote down lessons in order to relieve the burden imposed on teachers. They have produced several CD-ROMs containing their lessons and description of materials. The teamwork of Demal on theoretical ground and of Popeler in the classrooms has proved convincing to teachers, school administrations and the inspecting board of the Ministry of Education of the Communauté Française de Belgique.

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